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Generation of Compton harmonics by scattering linearly polarized light of arbitrary intensity from free electrons of arbitrary initial velocity

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Abstract. We present an analytic classical relativistic derivation for a general expression of the harmonic power generated per unit laboratory solid angle due to Compton scattering of plane wave, linearly polarized light of arbitrary intensity from free electrons moving initially with arbitrary velocity. We show graphically the generated frequency as a function of the coordinates of the observation point for several initial electron kinetic energies and light field intensities.

1. Introduction

It is well known that a free electron is not capable of absorbing or emitting a single photon due to the energy-momentum conservation condition. However, emission and absorption of two or more photons result, generally, in the recoil of the electron and the scattering of light. We have recently studied [1–3] the classical relativistic electron dynamics, including ponderomotive scattering and electron trajectories, in the presence of a plane wave, circularly polarized laser field of arbitrary intensity from a free electron initially moving with an arbitrary velocity. We have also made a systematic study of the related process of harmonic generation [4, 5]. These issues are of current interest in connection with laser-assisted fusion experiments [6], the design and operation of linear electron accelerators employing powerful lasers [7] and related problems.

In a classic 1970 paper, Sarachik and Schappert [8] presented a derivation for the nth harmonic power generated by scattering plane wave light from a single electron assumed to be initially *at rest* at the origin. Their analysis of this restricted problem was carried out in the reference frame in which the electron is *on average* at rest, the result was then Lorentz-transformed to the laboratory frame. The aim of this paper is to generalize the work of Sarachik and Schappert [8, 9] to the case of an electron initially moving with an *arbitrary* velocity. In this context, it should also be mentioned that the full QED problem was formulated a long time ago by Brown and Kibble [10].

In this paper, we present a classical, fully relativistic, analytic solution to the problem of harmonic production by scattering linearly polarized radiation from a free electron, again, without making any restrictions on the laser field intensity or the electron initial velocity. We carry out the whole analysis in the laboratory frame, thus eliminating all confusion and

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difficulties arising from the need to solve the problem in an oscillating frame (the average at rest frame) and then transform the results to the laboratory. The linear polarization case is more common, from a practical point of view, but yet is more involved from the calculational viewpoint, than the circular polarization problem.

This paper is organized as follows. The problem will be formulated in the next section. In section 3, we will present the full derivation of an expression for the nth harmonic power generated by the scattering process. We give a brief discussion of our results in section 4 and a summary in section 5.

2. Background

The plane wave, linearly polarized radiation field, frequency ω_0 and propagation vector $\boldsymbol{k} = (\omega_0/c)\hat{\boldsymbol{k}}$, will be modelled by the vector potential

$$\boldsymbol{A}(\eta) = \hat{\boldsymbol{\imath}} a \cos\eta \tag{1}$$

where a is a constant amplitude, $\eta = \omega_0 t - \boldsymbol{k}\cdot\boldsymbol{r}$ is the phase, t is the time, $\boldsymbol{r}$ is the position vector of the electron, and c is the speed of light. The initial velocity vector, scaled by the speed of light, will be given by

$$\boldsymbol{\beta}_0 = \beta_0(\hat{\boldsymbol{\imath}}\sin\theta_0 + \hat{\boldsymbol{k}}\cos\theta_0) \tag{2}$$

where θ_0 is the angle $\boldsymbol{\beta}_0$ makes with $\boldsymbol{k}$. We further let the unit vector $\hat{\boldsymbol{m}} = (m_1, m_2, m_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ point in the direction of observation of the scattered radiation, in a spherical polar coordinate system with origin at the point of intersection of the laser beam and the initial electron direction of motion.

The starting point for a systematic derivation of the scattered power is the following expression for the energy scattered per unit solid angle $\mathrm{d}\Omega$ and per unit frequency $\mathrm{d}\omega$, given here in the far-field approximation [11]

$$\frac{\mathrm{d}^2E}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{(e\omega)^2}{4\pi^2c^3}\left|\int_{-\infty}^{\infty}\hat{\boldsymbol{m}}\times\left(\hat{\boldsymbol{m}}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)\exp\left\{\mathrm{i}\omega\left[t-\frac{\hat{\boldsymbol{m}}\cdot\boldsymbol{r}(t)}{c}\right]\right\}\mathrm{d}t\right|^2. \tag{3}$$

We therefore need an expression for the electron trajectory, $\boldsymbol{r}(t)$ or equivalently $\boldsymbol{r}(\eta)$. An electron, of mass m_e, initially moving at the arbitrary velocity given by equation (2) in the presence of a plane wave radiation field follows a trajectory given by [4]

$$\begin{aligned}\boldsymbol{r}(\eta) = \boldsymbol{r}_0 &+ \frac{c}{\omega_0}\int_{\eta_0}^{\eta}\left[\frac{\gamma_0 m_\mathrm{e}c\boldsymbol{\beta}_0 + \frac{e}{c}\boldsymbol{A}(\eta')}{\gamma_0 m_\mathrm{e}c(1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0)}\right]\mathrm{d}\eta' \\ &+\hat{\boldsymbol{k}}\left(\frac{c}{\omega_0}\right)\int_{\eta_0}^{\eta}\left[\frac{\frac{1}{2}\left(\frac{e\boldsymbol{A}(\eta')}{\gamma_0 m_\mathrm{e}c^2}\right)^2 + \left(\frac{e\boldsymbol{A}(\eta')}{\gamma_0 m_\mathrm{e}c^2}\right)\cdot\boldsymbol{\beta}_0}{(1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0)^2}\right]\mathrm{d}\eta'.\end{aligned} \tag{4}$$

In parametric form, the electron trajectory in the linearly polarized field modelled by equation (1) may now be written down from equation (4) as:

$$x(\eta) = \frac{c}{\omega_0}(a_1\eta + b_1\sin\eta) \tag{5}$$

$$y(\eta) = 0 \tag{6}$$

$$z(\eta) = \frac{c}{\omega_0}\left(a_3\eta + b_3\sin\eta + \frac{c_3}{2}\sin 2\eta\right). \tag{7}$$

In these equations, η plays the role of a convenient parameter, and

$$a_1 = \frac{\beta_0 \sin\theta_0}{1-\beta_0\cos\theta_0} \tag{8}$$

$$b_1 = \frac{q/\gamma_0}{1-\beta_0\cos\theta_0} \tag{9}$$

$$a_3 = \frac{\beta_0 \cos\theta_0}{1-\beta_0\cos\theta_0} + \frac{(q/2\gamma_0)^2}{(1-\beta_0\cos\theta_0)^2} \tag{10}$$

$$b_3 = \frac{(q/\gamma_0)\beta_0 \sin\theta_0}{(1-\beta_0\cos\theta_0)^2} \tag{11}$$

$$c_3 = \frac{(q/2\gamma_0)^2}{(1-\beta_0\cos\theta_0)^2}. \tag{12}$$

Moreover, $q^2 = (ea/m_e c^2)^2$ is a dimensionless intensity parameter, and $\gamma_0 = (1-\beta_0^2)^{-1/2}$. Note that, in writing equations (5)–(7) from equation (4), η_0 and $\boldsymbol{r}_0$ have been dropped, the reason being that they enter into equation (3) for the scattered energy only through an unimportant phase factor. Note also that the trajectory is confined to the xz-plane, the plane containing the polarization vector and $\boldsymbol{\beta}_0$ [3].

3. Harmonic generation

With the parametric equations at our disposal, we may now write equation (3) as:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{(e\omega)^2}{4\pi^2 c^3}\{(1-m_1^2)(K_x)^2 - 2m_1 m_2 K_x K_z + (1-m_3^2)(K_z)^2\} \tag{13}$$

where we have changed the integration variable from t to η, and have taken

$$\boldsymbol{K} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\eta} \exp\left\{\mathrm{i}\frac{\omega}{\omega_0}\left[\eta + \frac{\omega_0}{c}[z - \hat{\boldsymbol{m}}\cdot\boldsymbol{r}(\eta)]\right]\right\} \mathrm{d}\eta. \tag{14}$$

From equation (14), and after lengthy algebra, we obtain the following expressions:

$$K_x = \frac{\pi c}{V}\sum_{n=-\infty}^{\infty}\{a_1 G_0^{(n)} + b_1 G_1^{(n)}\}\delta\left(\omega - n\frac{\omega_0}{V}\right) \tag{15}$$

$$K_z = \frac{\pi c}{V}\sum_{n=-\infty}^{\infty}\{a_3 G_0^{(n)} + b_3 G_1^{(n)} + c_3 G_2^{(n)}\}\delta\left(\omega - n\frac{\omega_0}{V}\right) \tag{16}$$

where

$$V = 1 - m_1 a_1 - (m_3 - 1)a_3 \tag{17}$$

and

$$X = (1-m_3)\frac{c_3}{2}\frac{\omega}{\omega_0} \tag{18}$$

$$Y = [m_1 b_1 + (m_3 - 1)b_3]\frac{\omega}{\omega_0} \tag{19}$$

and, yet ($s = 0, 1, 2$)

$$G_s^{(n)} = \sum_{\ell=-\infty}^{\infty} J_\ell(X)[J_{n+2\ell+s}(Y) + J_{n+2\ell-s}(Y)]. \tag{20}$$

The algebra leading to equations (15) and (16) involves the following. First, the trigonometric functions in $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ in equation (14) are expressed in exponential form. Secondly, the generating function of the ordinary Bessel functions

$$\mathrm{e}^{\mathrm{i}u\sin\xi} = \sum_{n=-\infty}^{\infty} J_n(u)\mathrm{e}^{\mathrm{i}n\xi} \tag{21}$$

is then used in part of the integrand. Thirdly, the integrations over η are carried out giving δ-functions. Finally, the dummy summation indices are changed such as to allow for extraction of a common δ-function.

We now simplify equations (15) and (16) further. To this end, we employ the well known recurrence relations of the ordinary Bessel functions to derive the following identity

$$G_0^{(n)} = \frac{Y}{n}G_1^{(n)} - \frac{2X}{n}G_2^{(n)}. \tag{22}$$

Using equation (22), we eliminate $G_0^{(n)}$ entirely from equations (15) and (16), which then become

$$K_x = \frac{\pi c}{V}\sum_{n=-\infty}^{\infty}\left\{\left(a_1\frac{Y}{n}+b_1\right)G_1^{(n)} - \left(2a_1\frac{X}{n}\right)G_2^{(n)}\right\}\delta\left(\omega - n\frac{\omega_0}{V}\right) \tag{23}$$

$$K_z = \frac{\pi c}{V}\sum_{n=-\infty}^{\infty}\left\{\left(a_3\frac{Y}{n}+b_3\right)G_1^{(n)} + \left(c_3 - 2a_3\frac{X}{n}\right)G_2^{(n)}\right\}\delta\left(\omega - n\frac{\omega_0}{V}\right). \tag{24}$$

We note at this point that, in view of the presence of the δ-function in equations (23) and (24) above, it follows that the Compton radiation is emitted only at the nth *harmonic* frequency:

$$\omega = \omega^{(n)} = \frac{n\omega_0}{1 - m_1a_1 - (m_3-1)a_3}. \tag{25}$$

With the help of equation (25), the definitions given above for X and Y, for example, become

$$X = n\left[\frac{(1-m_3)}{1 - m_1a_1 - (m_3-1)a_3}\right]\frac{c_3}{2} \tag{26}$$

$$Y = n\left[\frac{m_1b_1 + (m_3-1)b_3}{1 - m_1a_1 - (m_3-1)a_3}\right]. \tag{27}$$

Next, we transform the energy expression, equation (3), into an expression for the scattered power. The latter is defined by:

$$P = \lim_{T\to\infty}\frac{E}{T} \tag{28}$$

where T is a measure of time. To accomplish this, an integral representation for one of the δ-functions resulting from substituting equations (23) and (24) into equation (3) is used, whereby

$$\begin{aligned}\delta(\omega-\omega') &= \lim_{T\to\infty}\int_{-T/2}^{T/2}\mathrm{e}^{\mathrm{i}(\omega-\omega')t}\frac{\mathrm{d}t}{2\pi}\\ &= \frac{T}{2\pi} \qquad \text{only for } \omega = \omega'.\end{aligned} \tag{29}$$

Using the remaining δ-function, we then integrate the expression obtained from equation (3), after the operations implied by equations (28) and (29) have been carried out, over all frequencies in order to get the power scattered per unit laboratory solid angle. The result is the *total* scattered power, where *total* here means summed over all the harmonics from $n = 1$ to ∞; the terms corresponding to zero and negative values of n are dropped since frequencies can only be positive. We obtain the contribution to the total power from the nth harmonic by simply dropping the summation sign. Hence,

$$\begin{aligned}\frac{\mathrm{d}P^{(n)}}{\mathrm{d}\Omega} = \frac{(e\omega_0)^2}{8\pi c}\frac{n^2}{[1 - a_1\cos\alpha + 2a_3\sin^2(\theta/2)]^4} \\ \times\Bigg\{\Bigg[\sin^2\alpha\left(a_1\frac{Y}{n}+b_1\right)^2 - 2\cos\alpha\cos\theta\left(a_1\frac{Y}{n}+b_1\right)\left(a_3\frac{Y}{n}+b_3\right) \\ +\sin^2\theta\left(a_3\frac{Y}{n}+b_3\right)^2\Bigg](G_1^{(n)})^2 - \Bigg[4\sin^2\alpha\left(a_1\frac{Y}{n}+b_1\right)\left(a_1\frac{X}{n}\right) \\ +2\cos\alpha\cos\theta\left(a_1\frac{Y}{n}+b_1\right)\left(c_3-2a_3\frac{X}{n}\right) \\ -4\cos\alpha\cos\theta\left(a_3\frac{Y}{n}+b_3\right)\left(a_1\frac{X}{n}\right) \\ -2\sin^2\theta\left(a_3\frac{Y}{n}+b_3\right)\left(c_3-2a_3\frac{X}{n}\right)\Bigg]G_1^{(n)}G_2^{(n)} \\ +\Bigg[4\sin^2\alpha\left(a_1\frac{X}{n}\right)^2 + 4\cos\alpha\cos\theta\left(a_1\frac{X}{n}\right)\left(c_3-2a_3\frac{X}{n}\right) \\ +\sin^2\theta\left(c_3-2a_3\frac{X}{n}\right)^2\Bigg](G_2^{(n)})^2\Bigg\}\end{aligned} \tag{30}$$

where α is the angle between $\hat{\boldsymbol{m}}$ and the x-axis, i.e. $\cos\alpha = \sin\theta\cos\phi$. Equation (30), or equivalently equation (37) below, is the centrepiece of this paper. We now consider the limit of equation (30) as $\beta_0 \to 0$. In this limit, equations (8)–(12) yield

$$a_1 = b_3 = 0 \qquad b_1 = q \qquad \text{and} \qquad a_3 = c_3 = \frac{q^2}{4}. \tag{31}$$

Using this set of values for the coefficients in equations (17), (26) and (27), we obtain

$$\begin{aligned} V &= 1 + \tfrac{1}{2}q^2\sin^2(\theta/2) \qquad X = \frac{1}{4}nq^2\frac{\sin^2(\theta/2)}{1+\frac{1}{2}q^2\sin^2(\theta/2)} \\ Y &= \frac{nq\cos\alpha}{1+\frac{1}{2}q^2\sin^2(\theta/2)}.\end{aligned} \tag{32}$$

Using equations (31) and (32), we can easily show that $G_s^{(n)} \to F_s^n$ and that equation (30) reduces identically to

$$\begin{aligned}\frac{\mathrm{d}P^{(n)}}{\mathrm{d}\Omega} = \frac{(e\omega_0 q)^2}{8\pi c}\frac{n^2}{[1+\frac{1}{2}q^2\sin^2(\theta/2)]^4}\Bigg\{\left(1 - \frac{(1+\frac{1}{2}q^2)\cos^2\alpha}{[1+\frac{1}{2}q^2\sin^2(\theta/2)]^2}\right)(F_1^n)^2 \\ -\frac{1}{2}q\frac{\cos\alpha[\cos\theta - \frac{1}{2}q^2\sin^2(\theta/2)]}{[1+\frac{1}{2}q^2\sin^2(\theta/2)]^2}F_1^nF_2^n \\ +\frac{1}{16}q^2\frac{\sin^2\theta}{[1+\frac{1}{2}q^2\sin^2(\theta/2)]^2}(F_2^n)^2\Bigg\}\end{aligned} \tag{33}$$

where

$$F_s^n = \sum_{\ell=-\infty}^{\infty} J_\ell\left(\frac{1}{4}nq^2\frac{\sin^2(\theta/2)}{1+\frac{1}{2}q^2\sin^2(\theta/2)}\right)$$

$$\times \left\{ J_{2\ell+n+s}\left(\frac{nq\cos\alpha}{1+\frac{1}{2}q^2\sin^2(\theta/2)}\right) + J_{2\ell+n-s}\left(\frac{nq\cos\alpha}{1+\frac{1}{2}q^2\sin^2(\theta/2)}\right)\right\} \tag{34}$$

and where s is an integer equal to 1 or 2. This limited result was obtained by Sarachik and Schappert [8] many years ago for the nth harmonic power scattered into a unit laboratory solid angle by an electron initially at rest at the origin. Our analysis demonstrates that the average at rest frame employed in [8], in the case of an electron initially at rest, is justified.

We close by noting that equation (30), may be easily converted into an expression, for the harmonic power, in terms of the generalized Bessel functions $J(u, v)$. The generalized Bessel functions have always been associated with problems involving high-intensity linearly polarized light [12, 13]. It suffices, for our purposes in this work, to just recall the series representation of such functions in terms of the ordinary Bessel functions, namely

$$\begin{aligned} J_p(u, v) &= \sum_{k=-\infty}^{\infty} J_k(v) J_{p-2k}(u) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k J_k(v) J_{p+2k}(u) \\ &= \sum_{k=-\infty}^{\infty} J_k(-v) J_{p+2k}(u). \end{aligned} \tag{35}$$

In the second line of equation (35), we have let $k \to -k$, and in the third the parity property of the ordinary Bessel functions has been employed. Using equation (35), we obtain equation (20) in the form

$$G_s^{(n)} = J_{n+s}(Y, -X) + J_{n-s}(Y, -X). \tag{36}$$

In this context, equation (22) becomes one of the well known recurrence relations of the generalized Bessel functions [13]. Hence, in place of equation (30) we would have

$$\begin{aligned} \frac{\mathrm{d}P^{(n)}}{\mathrm{d}\Omega} = \frac{(e\omega_0)^2}{8\pi c} & \frac{n^2}{[1 - a_1\cos\alpha + 2a_3\sin^2(\theta/2)]^4} \\ & \times \{\sin^2\alpha (S_1^{(n)})^2 - 2\cos\alpha\sin\theta (S_1^{(n)} S_2^{(n)}) + \sin^2\theta (S_2^{(n)})^2\} \end{aligned} \tag{37}$$

where

$$S_1^{(n)} = \left(a_1\frac{Y}{n} + b_1\right) G_1^{(n)} - \left(2a_1\frac{X}{n}\right) G_2^{(n)} \tag{38}$$

$$S_2^{(n)} = \left(a_3\frac{Y}{n} + b_3\right) G_1^{(n)} + \left(c_3 - 2a_3\frac{X}{n}\right) G_2^{(n)}. \tag{39}$$

The set of equations (36)–(39) is equivalent to equations (20) and (30).

4. Discussion

Equation (25) gives the generated Compton frequencies that would be observed at the angular position (θ, ϕ). In figures 1 and 2, we plot the quantity $\frac{\omega}{n\omega_0}$ against the angular coordinates of the observation point. In both figures, the initial electron motion is in the direction corresponding to $\theta_0 = \pi/6$ and $\phi_0 = 0$, i.e. in the plane containing the field polarization and propagation directions. In figure 1, the laser field intensity is about 10^{18} W cm^{-2} ($q \approx 1$) and the electron's initial speed is such that: (a) $\gamma_0 = 10$ corresponding to a nearly 4.5 MeV electron, and (b) $\gamma_0 = 100$ or a ≈ 50 MeV electron. Note, first of all,

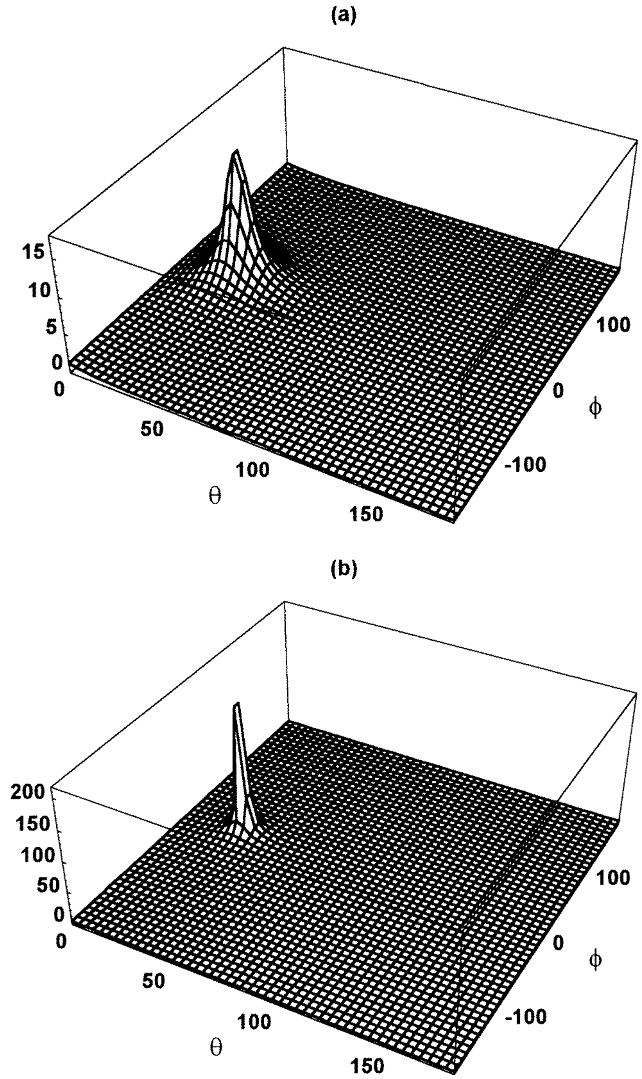

Figure 1. The quantity $\omega/n\omega_0$ (vertical axis) is shown here versus the angular coordinates θ and ϕ (both in degrees) of the observation point. The incident radiation has intensity $\approx 10^{18}$ W cm^{-2} ($q = 1$), and the electron initially moves in the plane containing the laser field propagation and polarization directions ($\theta_0 = \pi/6$ with $\boldsymbol{k}$). The initial electron energy is approximately: (a) 4.5 MeV ($\gamma_0 = 10$), and (b) 50 MeV ($\gamma_0 = 100$).

that the generated frequencies are confined to a narrow cone around the direction of initial electron motion, as expected, and that the cone becomes narrower with increasing electron initial speed. Secondly, the magnitude of the generated frequency increases with increasing initial electron speed.

In figure 2, we plot the same quantity, but for: (a) $q = 10, \gamma_0 = 100$, and (b) $q = 100, \gamma_0 = 100$. In this figure, the effect on the generated frequency of increasing the laser field intensity is shown. As q increases by one order of magnitude (corresponding to a two-order-of-magnitude increase in the intensity) from (a) to (b) the maximum frequency

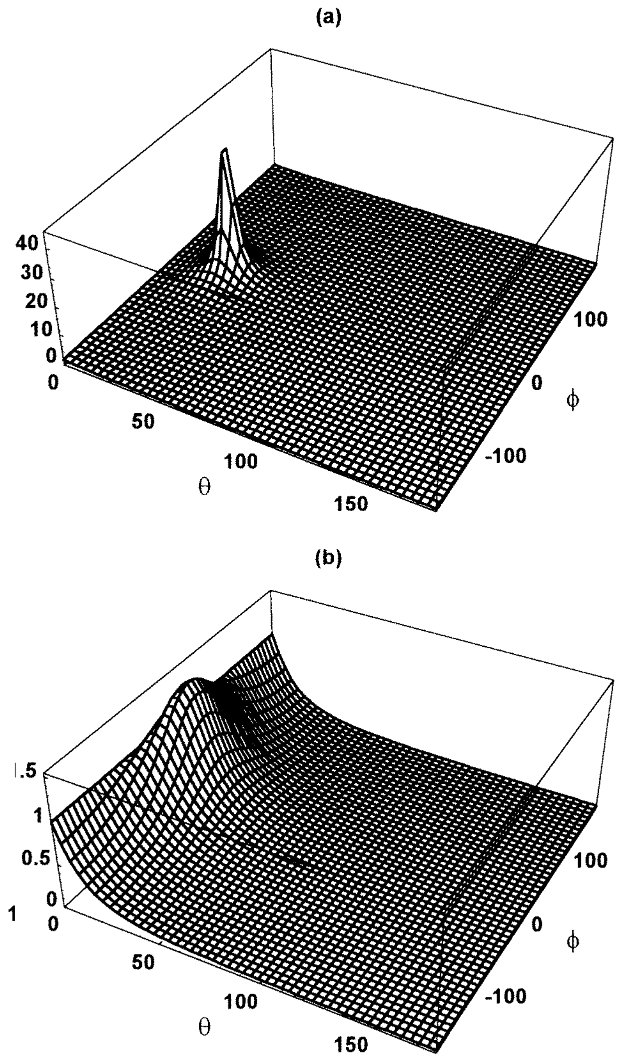

Figure 2. The same as figure 1, but for $\gamma_0 = 100$, and: (*a*) $q = 10$, and (*b*) $q = 100$.

generated decreases while, at the same time, lower frequencies show up far away from the initial electron direction of motion, i.e. scattered radiation is no longer confined to a narrow cone about the electron's initial direction of motion.

5. Summary

In conclusion, we have considered the (Compton) scattering of linearly polarized light off relativistic electrons. The situation we chose to study is general in the sense of involving arbitrary initial electron energy and momentum and arbitrary laser intensity. The angular distribution in the laboratory of the power emitted per unit solid angle into the Compton harmonic of order n has been found exactly analytically. Two equivalent expressions for this quantity have been obtained, one, equation (30), in terms of the ordinary Bessel functions, the other, equation (37), involving the generalized Bessel functions. The expression derived many years ago for the restricted problem of an electron initially at rest at the origin has been shown to follow exactly from our main result in the appropriate limit, $\beta_0 \to 0$. By so doing, we have rigorously demonstrated that the average at rest frame employed by Sarachik and Schappert in this case is justified.

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